2) The observer is at the same distance, but at an angle $\phi = 30$ deg from the rotor plane, $(x, y, z) = (-480, 0, 240)$.

The agreement between theoretical and Kirchhoff solutions is very good for both cases with error less than 1% for peak values. The discrepancy at the beginning is due to the sudden jump at the solution there. Case 2 has a lower signal, as expected, since the thickness noise is maximum at the rotor plane.

This simple example shows the viability of the method for three-dimensional helicopter noise calculations. The Kirchhoff method is not limited to the sonic cylinder, and the observer can be out of the rotor plane. Solutions can also be obtained for a moving rotor. This powerful method can be utilized for the development of a set of simple portable Kirchhoff subroutines for the calculation of the far-field noise using the input given from any aerodynamic near/midfield code.

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New Series Expansion Method for the Solution of the Falkner-Skan Equation

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Introduction

THE Falkner-Skan equation is a celebrated equation in fluid mechanics. Its solutions are well known as similar solutions. The study of the similar solutions in the past has contributed greatly to our understanding of the mechanism which governs the process of heat, mass, and momentum transfer through laminar boundary layers. It is of interest to note that the similar solutions are also required in several series solutions of partial differential equations, where they usually appear as the first term. Similar solutions are important because they also serve as the foundation for a number of more general methods for boundary-layer analysis used for estimating transfer rates for both similar and nonsimilar boundary layers. Because of the theoretical and practical importance of the Falkner-Skan equation as stated above and because no closed-form solution is known, a number of numerical solutions for many discrete values of β have been tabulated and many scientists have worked on it, such as Cebeci and Keller,[1] Smith,[2] and Evans,[3] among others. However, all of their solutions are in discrete form. Recently, Aziz and Na[4] described a new approach for this problem. They explored a regular perturbation expansion for f as a power series in β for this problem. Compared to the previous methods, the power-series method is similar both in conception and computation. But the resulting series is found to converge only for very small values of β. To improve the range of applicability and accuracy, the Shanks transformation is applied repeatedly. This process is not only inconvenient but also inaccurate. Under the motivation of this approach and the research of Van Dyke,[5,6] we derive a new series expansion method for this problem. Four series for $f''(0)$ and four series for δ_1 originating at $\beta_0 = 2, 1, 0.3, 0$, respectively, have been worked out. To overcome the difficulty near separation, a common term is constructed and used to estimate the remainder of the series originating at $\beta_0 = 0$. The surprising agreement of the results with the well-known numerical solutions shows the high accuracy of the present method.

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Mathematical Model

The celebrated Falkner-Skan equation in fluid mechanics is of the form

$$f''' + ff'' + \beta(1 - f'^2) = 0 \tag{1}$$

with boundary conditions

$$f(0) = f'(0) = 0, \qquad f'(\infty) = 1 \tag{2}$$

Substituting $t = \beta_0 - \beta$ into Eq. (1), we have

$$f''' + ff'' + \beta_0(1 - f'^2) = t(1 - f'^2) \tag{3}$$

Assuming

$$f = \sum_{n=0}^{\infty} t^n f_n \tag{4}$$

we have

$$f_0''' + f_0 f_0'' + \beta_0(1 - f_0'^2) = 0 \tag{5}$$

$$f_0(0) = f_0'(0) = 0, \qquad f_0'(\infty) = 1 \tag{6}$$

$$f_n''' + f_0 f_n'' - 2\beta_0 f_0' f_n' + f_0'' f_n = \delta_{1n} - \sum_{r=1}^{n} f_{r-1}' f_{n-r}' - \sum_{r=1}^{n-1} f_r f_{n-r}'' + \beta_0 \sum_{r=1}^{n-1} f_r' f_{n-r}' \tag{7}$$

$$f_n(0) = f_n'(0) = 0, \qquad f_n'(\infty) = 0 \tag{8}$$

where $n = 1, 2, 3, \ldots$ and δ_{1n} is the well-known Kronecker delta.

The skin-friction factor is

$$f''(0) = \sum_{n=0}^{\infty} f_n''(0) t^n = \sum_{n=0}^{\infty} a_n t^n \tag{9}$$

The displacement-thickness factor is

$$\delta_1 = \int_0^\infty (1 - f')\,\mathrm{d}\eta = \lim_{\eta\to\infty}\left\{\eta - \sum_{n=0}^{\infty} f_n t^n\right\} = \sum_{n=0}^{\infty} b_n t^n \tag{10}$$

The momentum-thickness factor is related to δ_1 and $f''(0)$ in the form

$$f''(0) = \beta\delta_1 + (1+\beta)\delta_2 \tag{11}$$

Series Solution

For four given values of β_0 (i.e., the origin of the series expansion), the numerical solutions of Eqs. (5-8) and Eq. (10) give us the corresponding a_n and b_n as shown in Tables 1 and 2, together with the range of application.

It is found that

$$[f''(0)]^2 = \left[\sum_{n=0}^{\infty} a_n t^n\right]^2 = \sum_{n=0}^{\infty} c_n t^n \tag{12}$$

$$D_1^2 = \left[\partial_{1\text{sep}} - \sum_{n=0}^{\infty} b_n t^n\right]^2 = \sum_{n=0}^{\infty} d_n t^n \tag{13}$$

($D_1 = \delta_{1\text{sep}} - \delta_1 = 2.35884570 - \delta_1$). This is somewhat better from the viewpoint of convergency. Also,

$$c_0 = a_0^2, \quad c_1 = 2a_0 a_1, \quad c_2 = a_1^2 + 2a_0 a_2, \ldots \tag{14}$$

$$d_0 = b_0^{-2}, \quad d_1 = -2\bar{b}_0 b_1, \quad d_2 = b_1^2 - 2b_0 b_2, \ldots \tag{15}$$

($\bar{b}_0 = \partial_{1\text{sep}} - b_0$). The reason for this occurrence is that the function $[f''(0)]^2$ or D_1^2 with respect to β is closer to a straight line (see Fig. 1).

The corresponding c_n and d_n in Eqs. (12) and (13) are shown in Tables 3 and 4 together with the range of application. In the case of $\beta_0 = 0$, the corresponding series cannot cover the whole range of β near separation. To overcome this difficulty, we introduce the common-term construction method presented in the first author's previous paper.[7] From the numerical solution, we found that separation occurs at $\beta_{\text{sep}} =$

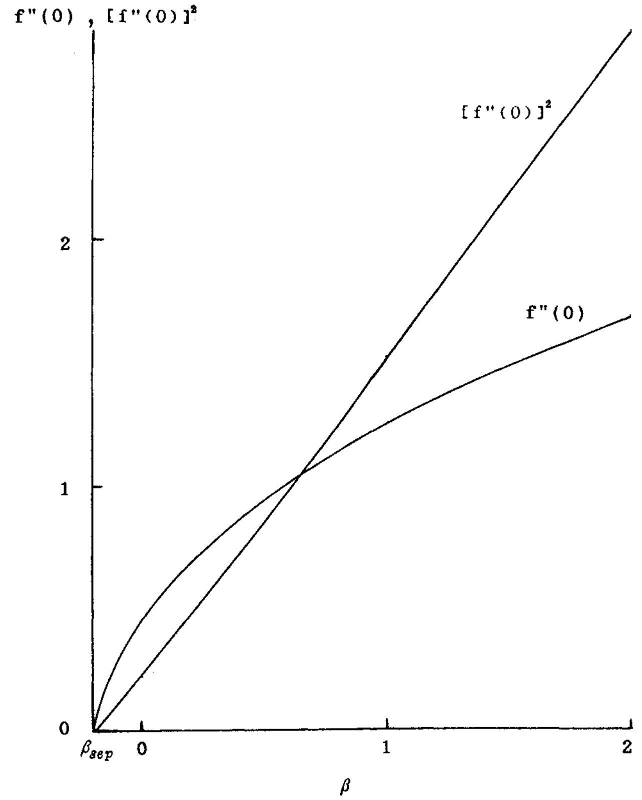

Fig. 1 $[f''(0)]^2$ **is closer to a straight line.**

Table 1 Coefficients of Eq. (9)

a_n	$\beta_0 = 2$	$\beta_0 = 1$	$\beta_0 = 0.3$	$\beta_0 = 0$
a_0	1.68721817	1.23258767	0.77475458	0.46959999
a_1	−0.39412582	−0.53671450	−0.83450727	−1.2989291
a_2	−0.04560122	−0.11383635	−0.41410405	−1.5220060
a_3	−0.01046873	−0.04743228	−0.39918958	−3.5619998
a_4	−0.00298461	−0.02442445	−0.47868911	−10.666717
a_5	−0.00094807	−0.01399464	−0.64602689	−36.434664
a_6	−0.00032138	−0.00856527	−0.94104246	−134.81195
a_7	−0.00011379	−0.00548803	−1.4459841	−525.87926
a_8	−0.00004158	−0.00363890	−2.3103660	−2129.2511
a_9	——	−0.00247847	−3.8019753	−8862.9806
a_{10}	——	−0.00172512	−6.4014576	−37688.211
Range	$2.7<\beta<1.3$	$1.5<\beta<0.5$	$0.5<\beta<0.1$	$0.1<\beta<-0.1$

Table 2 Coefficients of Eq. (10)

b_n	$\beta_0 = 2$	$\beta_0 = 1$	$\beta_0 = 0.3$	$\beta_0 = 0$
b_0	0.4974337	0.6479005	0.9109938	1.217806
b_1	0.1018692	0.2265274	0.6479480	1.675580
b_2	0.0314322	0.1203543	0.7264755	4.004006
b_3	0.0108246	0.0719647	0.9465300	11.88956
b_4	0.0039317	0.0457411	1.345097	40.07173
b_5	0.0014753	0.0302509	2.027941	146.4553
b_6	0.0005663	0.0205946	3.191907	565.5123
b_7	0.0002211	0.0143403	5.189857	2270.955
b_8	0.0000875	0.0101688	8.653623	9389.892
b_9	——	0.0073203	14.72015	39710.43
b_{10}	——	0.0053370	25.44753	170972.8
Range	$2.8<\beta<1.2$	$1.6<\beta<0.5$	$0.55<\beta<0.05$	$0.1<\beta<-0.1$

Table 3 Coefficients of Eq. (12)

c_n	$\beta_0 = 2$	$\beta_0 = 1$	$\beta_0 = 0.3$	$\beta_0 = 0$
c_0	2.84670515	1.51927233	0.60024466	0.22052415
c_1	−1.3299525	−1.3230953	−1.2930767	−1.2199541
c_2	0.00145674	0.00743590	0.05474436	0.25774872
c_3	0.00061917	0.00526636	0.07259776	0.60852737
c_4	0.00026011	0.00366334	0.09600222	1.5518864
c_5	0.00010820	0.00251771	0.12852654	4.3339467
c_6	0.00004464	0.00171795	0.17588087	13.194151
c_7	0.00001829	0.00116847	0.24726796	43.213731
c_8	0.00000746	0.00079472	0.35773195	150.07551
c_9	——	0.00054186	0.53221545	545.84017
c_{10}	——	0.00037109	0.81262656	2059.3134
Range	$2.8\le\beta\le1.2$	$1.65\le\beta\le0.5$	$0.55\le\beta\le0.05$	$0.1\le\beta\le-0.1$

Table 4 Coefficients of Eq. (13)

d_n	$\beta_0 = 2$	$\beta_0 = 1$	$\beta_0 = 0.3$	$\beta_0 = 0$
d_0	3.4648547	2.9273336	2.0962751	1.3043127
d_1	−0.3792412	−0.7751519	−1.876266	−3.827244
d_2	−0.1066391	−0.3605246	−1.683821	−6.338100
d_3	−0.0338940	−0.1917283	−1.799434	−13.73923
d_4	−0.0114435	−0.1094318	−2.140630	−35.65317
d_5	−0.0040109	−0.0654695	−2.753949	−105.0244
d_6	−0.0014433	−0.0405780	−3.764539	−338.6521
d_7	−0.0005299	−0.0258753	−5.399062	−1166.347
d_8	−0.0001977	−0.0168962	−8.046830	−4420.452
d_9	——	−0.0112589	−12.37238	−15866.10
d_{10}	——	−0.0076351	−19.51533	−61481.17
Range	$2.8\le\beta\le1.2$	$1.6\le\beta\le0.5$	$0.55\le\beta\le0.05$	$0.1\le\beta\le-0.1$

Table 5 Comparison of values of $f''(0)$

β	Smith[2]	Aziz and Na[4]	Present
2.0	1.687218	1.687516	1.687218
1.6	1.521514	1.521689	1.521514
1.2	1.335722	1.335793	1.335721
1.0	1.232588	1.232623	1.232588
0.8	1.120268	1.120280	1.120268
0.6	0.995836	0.995837	0.995836
0.4	0.854421	0.854418	0.854421
0.2	0.686708	0.686706	0.686708
0.1	0.587035	0.587034	0.587035
0.05	0.531130	0.531129	0.531130
0.00	0.469600	0.469600	0.469600
−0.05	0.400323	0.400322	0.400323
−0.10	0.319270	0.319266	0.319270
−0.14	0.239736	0.239724	0.239736
−0.16	0.190780	0.190758	0.190780
−0.18	0.128636	0.128615	0.128636
−0.19	0.085700	0.085840	0.085700
−0.195	0.055172	0.056027	0.055172

Table 6 Comparison of values of ∂_1 and ∂_2

	∂_1		∂_2	
β	Evans[3]	present	Evans[3]	present
2.0	0.49743	0.49743	0.23079	0.23078
1.6	0.54402	0.54402	0.25042	0.25042
1.2	0.60689	0.60690	0.27612	0.27611
1.0	0.64790	0.64790	0.29234	0.29234
0.8	0.69868	0.69868	0.31185	0.31185
0.6	0.76397	0.76397	0.33591	0.33591
0.5	0.80455	0.80455	0.35027	0.35027
0.4	0.85264	0.85263	0.36669	0.36669
0.3	0.91099	0.91099	0.38574	0.38574
0.2	0.98416	0.98416	0.40823	0.40823
0.1	1.08032	1.08032	0.43546	0.43546
0.0	1.21677	1.21678	0.46960	0.46960
−0.10	1.44270	1.44270	0.51504	0.51504
−0.14	—	1.59590	—	0.53856
−0.16	—	1.70665	—	0.55219
−0.18	—	1.87157	—	0.56771
−0.19	—	2.00676	—	0.57625
−0.195	—	2.11704	—	0.58136

−0.198837735, where we have $f''(0) = 0$ and $\delta_1 = 2.35884570$. The corresponding series can then be written as

$$[f''(0)]^2 = (1-\bar{t})(0.22052415 - 0.02204877\bar{t} - 0.01185830\bar{t}^2 - 0.00707446\bar{t}^3 - 0.00464866\bar{t}^4 - 0.00330163\bar{t}^5 - 0.00248622\bar{t}^6 - 0.00195520\bar{t}^7 - 0.00158851\bar{t}^8 - 0.00132332\bar{t}^9 - 0.00112439\bar{t}^{10} + R_1), \quad \beta_{sep} \le \beta \le 0.1 \tag{16}$$

$$D_1^2 = (1-\bar{t})(1.3043127 + 0.5433121\bar{t} + 0.2927262\bar{t}^2 + 0.1847175\bar{t}^3 + 0.1289869\bar{t}^4 + 0.0963444\bar{t}^5 + 0.0754155\bar{t}^6 + 0.0610830\bar{t}^7 + 0.0507709\bar{t}^8 + 0.0430626\bar{t}^9 + 0.0371230\bar{t}^{10} + R_2), \quad \beta_{sep} \le \beta \le 0.1 \tag{17}$$

where $\bar{t} = t/0.198837735$ and R_1 and R_2 are the remainders of the series.

It is found that the last few terms in Eq. (16) may be approximated by a so-called "common term"

$$c_n\bar{t}^n = k_1\bar{t}^n(n + a_1)^{s_1} \tag{18}$$

where constants k_1, a_1, and s_1 may be determined by three nodal values (say, c_7, c_9, and c_{10}) to be $s_1 = -1.50383$, $a_1 = -0.25172867$, and $k_1 = -0.03452676$. Similarly, the last few terms in Eq. (17) may be approximated by a common term

$$d_n\bar{t}^n = k_2\bar{t}^n(n + a_2)^{s_2} \tag{19}$$

where $s_2 = -1.5016$, $a_2 = 0.6296443$, and $k_2 = 1.291484$. The remainder of the series can then be written as

$$R_1 = k_1 \sum_{n=11}^{\infty} \bar{t}^n(n + a_1)^{s_1} \tag{20}$$

$$R_2 = k_2 \sum_{n=11}^{\infty} \bar{t}^n(n + a_2)^{s_2} \tag{21}$$

Within the range $-0.1<\beta<2.7$, the values of $f''(0)$ and δ_1 may be obtained directly from the corresponding series of Eqs. (12) and (13). Although in the range $\beta_{sep}<\beta<-0.1$, the series given by Eqs. (16) and (17) must be used. As for δ_2, it may be calculated easily from Eq. (11) when $f''(0)$ and δ_1 are known.

The results for 18 values of β within the range $\beta_{sep}<\beta\ll 2$ given in Tables 5 and 6 show surprising agreement with the well-known numerical results of Smith[2] and Evans.[3]

Conclusions

Instead of the one-series expansion originated at $\beta_0 = 0$, four series with different origins developed in this paper cover almost the whole practical interest range of β.

The series for $[f''(0)]^2$ derived in this Note is somewhat better than the commonly used series for $f''(0)$ from the viewpoint of convergency, especially for small values of β. Similarly, the series for D_1^2 is better than the series for δ_1.

The series for $[f''(0)]^2$ and D_1^2 with a factor $(1-\bar{t})$ derived in this Note satisfy the conditions $\bar{t} = 1$, $f''(0) = 0$, and $D_1 = 0$ at the point of separation automatically.

By means of the information provided by the known terms (say, $c_7 - c_{10}$ or $d_7 - d_{10}$), the common term is constructed and used to estimate the remainder of the series.

Four series for $[f''(0)]^2$, four series for D_1^2, together with a simple exact relation between $f''(0)$ and δ_1 and δ_2, may be taken as a "quasianalytical solution" of the Falkner-Skan equation and thus lead the calculation to a very simple work.

Checking the accuracy of the results obtained in this Note, we believe that the present method is a very satisfactory one. It is accurate to six digits for $f''(0)$ and five digits for δ_1 and δ_2.

The new idea presented in this Note and outlined above may be probably applied to any other problems of a similar nature.

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